B-Spline Interpolation of Soil Water Characteristic Data

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Interpolation von Matrixpotenzial-Wassergehaltsbeziehungen mit B-Splines

1 Introduction

Soil water characteristic data are commonly measured for soil physical applications. To obtain values between measurements, empirical functions are typically fitted to the data, e.g. the ones of VAN GENUCHTEN et al. (1991) or BROOKS and COREY. However, they assume a certain shape that might not be appropriate for every data set. KASTANEK and NIELSEN (2001) presented a computer program for the interpolation of soil water characteristic data with natural cubic spline functions. In addition, the use of so-called "virtual data-points" enables the user to perform approximation of these data by hand. In some cases, this was considered necessary since cubic spline functions can exhibit significant oscillations between data points.

B-spline curves are piecewise polynomial parametric curves commonly used in computer graphics. Some of their features could be of advantage with regard to the interpolation of soil water characteristic data, such as the local support property.

Objective of this paper was to test the capabilities of the technique of B-spline curves to interpolate soil water char-

Zusammenfassung

Die Matrixpotenzial-Wassergehaltsbeziehung ist eine Grundgröße für jegliche Berechnung von Wasserbewegung und anderen Größen des Wasserhaushalts. Werte zwischen den Messpunkten können durch Approximationen oder Interpolationen berechnet werden. In der vorliegenden Arbeit wurde der mathematische Algorithmus der B-Spline Interpolation und dessen Anwendung auf Matrixpotenzial-Wassergehaltsbeziehungen beschrieben. Ein Computerprogramm zur Interpolation mit Hilfe von B-Splines und natürlichen kubischen Splines wurde in JAVA 2 implementiert. Splines bieten vor allem dann eine hilfreiche Alternative, wenn es nicht möglich ist, empirische Gleichungen wie die von VAN GENUCHTEN zu approximieren. Im Vergleich zu natürlichen kubischen Splines verhalten sich uniforme B-Splines zweiter und dritter Ordnung zwischen den Datenpunkten weniger oszillatorisch. Trotzdem garantieren sie nicht, dass die Interpolationskurve der Monotonie der Daten folgt. Die lokale Veränderbarkeit von B-Splines, ohne den Rest der Kurve zu beeinflussen, kann hier genutzt werden.

Schlagworte: Wassergehalt, Boden, B-Splines, Wasserbewegung, Matrixpotenzial.

Summary

Soil water characteristic data are of fundamental importance for all calculations of soil water movement. Values between measurements can be obtained by curve approximation or interpolation techniques. In this paper, the mathematical algorithm of B-spline interpolation is described and its use for interpolation of soil water characteristic data is demonstrated. A computer program for interpolation with B-splines and natural cubic splines was implemented in JAVA 2. In cases were VAN GENUCHTEN equations cannot be fitted to the data with sufficient accuracy, splines provide a helpful alternative. In comparison to natural cubic splines, uniform B-splines of degree 2 and 3 tend to exhibit less oscillations between data points. However, they do not guarantee to follow the monotony of the data in all cases. The local modification property of B-splines can be used to adjust segments of the curve while the rest of the curve stays unchanged.

Key words: Soil water, B-Spline, Matric potential head.

acteristic data. A computer program for interpolation with natural cubic splines and B-splines of various degrees and parametrizations was implemented in JAVA 2. Moving control points results in local changes of the shape of the curve (local support property). This feature was implemented in the JAVA 2 program allowing control points to be moved by mouse drags. This concept is similar to the concept of the virtual data points used by KASTANEK and NIELSEN (2001). It is different by the fact that the curve can be modified in one segment while the rest of the curve stays unchanged. Results were compared to the results of interpolation with natural cubic spline functions as well as curve fitting with the VAN GENUCHTEN equation (VAN GENUCHTEN et al. 1991).

2 Material and Methods

2.1 Data Sets

The data sets for the soil water characteristics were obtained from soil samples of a loamy silt from Mistelbach, 60 km north of Vienna. They are part of an investigation on a long term experimental field (FWF project P15329), where the influence of different tillage practices on soil quality is investigated. Three tillage systems are compared: Conventional tillage (CV), conservation tillage systems with cover crop during winter period (CS) and no-tillage systems with cover crop during winter period (NT). In fall 2002, after the harvest of mais, undisturbed samples where taken in four depths (0, 10, 25 and 50 cm) and two replicates for each treatment, resulting in a total of 24 data sets. Each value of each data set is again a mean of three replications. Samples were analyzed according to the Austrian standard ÖNORM L1063 (1988). Data sets were analyzed and cleaned from obvious errors before interpolation.

2.2 Data Representation

Whether soil water characteristic data are referred to as θ(h) or h(θ) has an influence on the interpolation result in most cases (except B-splines with uniform parametrization). In the following, the common θ(h) notation will be used (VAN GENUCHTEN et al. 1991).

Matric potential head of soil water characteristics is generally presented in both linear and log scale. Because transformation of zero matric potential head into log scale is problematic, all interpolations were performed using original data values. In graph presentations, both scales were used.

2.3 Parametric B-Spline Curves – Theory

Natural cubic splines and VAN GENUCHTEN's equation are described in KASTANEK and NIELSEN (2001). In the following, we describe in detail the theory of parameteric B-spline curves.

B-spline curves are piecewise polynomial parametric curves of degree p. A B-spline curve with $n+1$ control points is defined as

$$B_p(u) = \sum_{i=0}^{n} N_{i,p}(u)P_i, \quad \text{for} \quad u \in \mathbb{R}, \tag{1}$$

where $p \in \mathbb{N}$ is the degree of the B-spline curve, $N_{i,p} \in \mathbb{R}$ are the B-spline basis functions and $P_i \in \mathrm{R}^2$ are the associated control points.

The B-Spline basis functions can be defined recursively by the equation of COX-DE BOOR, Eq. (2),

$$\begin{aligned} N_{i,0}(u) &= \begin{cases} 1, & u_i \le u \le u_{i+1}, \\ 0, & \text{otherwise}, \end{cases} \\ N_{i,p}(u) &= \frac{u-u_i}{u_{i+p}-u_i} N_{i,p-1}(u) + \frac{u_{i+p+1}-u}{u_{i+p+1}-u_{i+1}} N_{i+1,p-1}(u). \end{aligned} \tag{2}$$

Significant features of B-splines $B_p(u)$ are (DE BOOR 1978):

1. $N_{i,p}(u)$ is a polynomial of order p in u,
2. For all i, p and u is $N_{i,p}(u)$ non-negative,
3. $N_{i,p}(u)$ is equal to zero except for the interval $[u_i, u_{i+1})$ (local support),
4. On the interval $[u_i, u_{i+1})$, only $p+1$ basis functions of order p are not equal zero, namely, $N_{i-p,p}(u)$, $N_{i-p+1,p}(u)$, ..., $N_{i,p}(u)$,
5. $N_{i,p}(u) \in C^{p-k}$ at knots with multiplicity k.

The third property can be used to modify interpolation curves of soil water characteristics locally while the rest of the curve stays unchanged.

2.3.1 *Global Curve Interpolation*

We wish to find a B-spline curve of degree p that passes through $n+1$ given data points $d_k := \begin{pmatrix} x_k \\ y_k \end{pmatrix}$, $k = 0, 1, ..., n$.

From these, the curve parameters $t_0, t_1, ..., t_n$ and the knot vector $U = (u_0, u_1, ..., u_m)$ have to be determined, with $m=n+p+1$.

Common parametrization methods are the uniformly spaced method, the cord length method and the centripetal method. We define the cord length method according to SPÄTH (1983):

$$
\begin{aligned}
t_0 &= 0, \\
t_k &= t_{k-1} + dist_{k-1}, \ \text{ for } k = 1, ..., n, \\
dist_k &= \sqrt{(\nabla x)^2 + (\nabla y)^2}, \ \text{ for } k = 0, ..., n-1.
\end{aligned} \tag{3}
$$

where $\nabla x_k = x_{k-1} - x_k, \nabla y_k = y_{k-1} - y_k$.

The uniformly spaced method is defined by dividing the domain $[x_0, x_n]$ into equal subintervals,

$$
\begin{aligned}
t_0 &= x_0, \\
t_k &= k\frac{x_n - x_0}{n}, \ \text{ for } 1 \leq k \leq n-1, \\
t_n &= x_n.
\end{aligned} \tag{4}
$$

The behavior of a curve based on this parametrization could be described by a car which drives in such a way that the time spent between points is constant. If the points are far enough away, the speed is relatively fast and might cause problems around sharp corners. The cord length parametrization shown in Eq. (3) can also cause problems around sharp corners.

The centripetal method is given by

$$
\begin{aligned}
t_0, &= 0, \\
t_k, &= t_{k-1} + (dist_{k-1})^{\frac{1}{2}}, \ \text{ for } k = 1, 2, ..., n, \\
dist_k &= \sqrt{(\nabla x)^2 + (\nabla y)^2}, \ \text{ for} k = 0, 1, ..., n-1,
\end{aligned} \tag{5}
$$

where $\nabla x_k = x_{k-1} - x_k, \nabla y_k = y_{k-1} - y_k$.

B-spline curves are defined in terms of *u*. The interval between two elements of the knot vector *U* in which *u* lies determines which control points determine the shape of the curve. There are several methods to determine the *m+p+1* knots. Knots can also be uniformly spaced, such a knot vector does not need knowledge of the parameters. However, this method is not recommended when used with the cord length method for global interpolation, because the resulting system of linear equations would be singular. Another way for generating knot vectors is:

$$
\begin{aligned}
u_0 &= u_1 = ... = u_p = t_0, \\
u_{j+p} &= \frac{1}{p}\sum_{i=j}^{j+p-1} t_i, \ \text{ for } j = 1, 2, .., n-p, \\
u_{m-p} &= u_{m-p+1} = ... = u_m = t_n.
\end{aligned} \tag{6}
$$

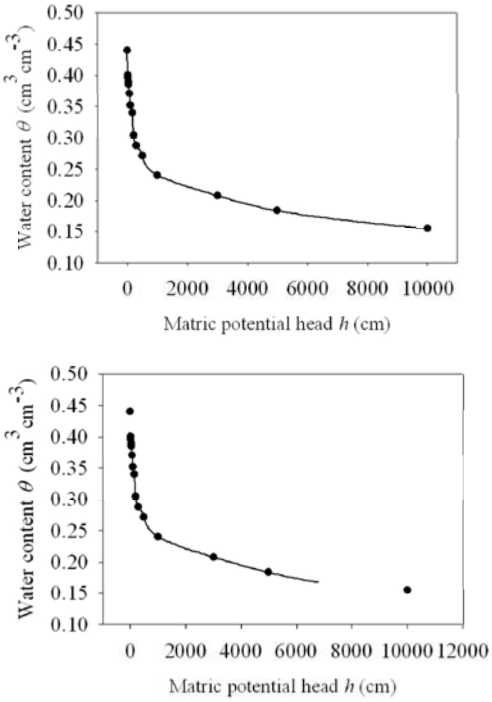

Figure 1: Cubic B-Spline interpolation (a) clamped and (b) unclamped

Abbildung 1: Interpolation mit kubischen Splines (a) eingespannt (b) nicht eingespannt.

Knot vectors where the first and last knot have multiplicity *p+1* are called clamped, the corresponding B-spline curve is tangent to the first and the last legs at the first and last control points. B-spline curves calculated from "un-clamped" vectors do not pass through the first and last data point and will therefore not be considered for interpolation of soil water characteristic data (see Figure 1).

Let N be a $(n+1) \times (n+1)$ matrix containing the values of the B-spline basis functions for the parameter values calculated with Eq. (2),

$$
N = \begin{pmatrix} N_{0,p}(t_0) & N_{1,p}(t_0) & \dots & N_{n,p}(t_0) \\ N_{0,p}(t_1) & N_{1,p}(t_1) & \dots & N_{n,p}(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ N_{0,p}(t_n) & N_{1,p}(t_n) & \dots & N_{n,p}(t_n) \end{pmatrix}. \tag{7}
$$

Let the given data points and the yet unknown control points be presented in matrix form

$$
D = \begin{pmatrix} d_{01} & d_{02} \\ d_{11} & d_{12} \\ \vdots & \vdots \\ d_{n1} & d_{n2} \end{pmatrix}, \ P = \begin{pmatrix} P_{01} & P_{02} \\ P_{11} & P_{12} \\ \vdots & \vdots \\ P_{n1} & P_{n2} \end{pmatrix}. \tag{8}
$$

Then P can be obtained from the system of equations

$$D = NP. \tag{9}$$

Knowing control points and knot vector allows to calculate points on the B-spline curve between the data points using DE BOOR's algorithm. As input, it needs a $u \in \mathbb{R}$ that lies between two elements of the knot vector U, output is a point on the curve $B_p(u)$. The interval in which u lies determines the $p+1$ control points that influence this particular curve segment.

3 Results and Discussion

3.1 Selection of B-spline type

Figure 2 shows how the three methods of parametrization perform on selected soil water characteristic data at degrees 2–4. The uniform and centripetal methods of degree 2 and 3 follow the monotony of the data most readily, whereas other methods showed significant oscillations between data points. Uniform B-splines of degree 2 and 3 were selected for subsequent interpolation of all 24 data sets.

3.2 Comparison with Van Genuchten Approximation and Natural Cubic Spline Interpolation Curves

Among the 24 considered data sets, two main cases were found were VAN GENUCHTEN's equation could not be well fitted. It happened when the data suggested that the curve was non-convex at a certain segment where VAN GENUCHTEN's equation could not follow. In other cases, VAN GENUCHTEN's equation under- or overestimated the water content at the highest or lowest matric potential values (Figure 3).

Empirical equations such as van Genuchten's are often easier to use for purposes such as modeling water movement. However, it might not be possible that this equation can be fitted to a particular soil water characteristic data set with sufficient accuracy. In this case, splines can be a helpful alternative.

Natural cubic splines exhibited significant oscillations in many of the 24 data sets (Figure 5). For interpolation of soil moisture characteristic data, their results are often inapplicable. As a solution to this problem, KASTANEK and NIELSEN (2001) suggested their concept of "virtual data points". B-splines and their local support property offer another convenient alternative.

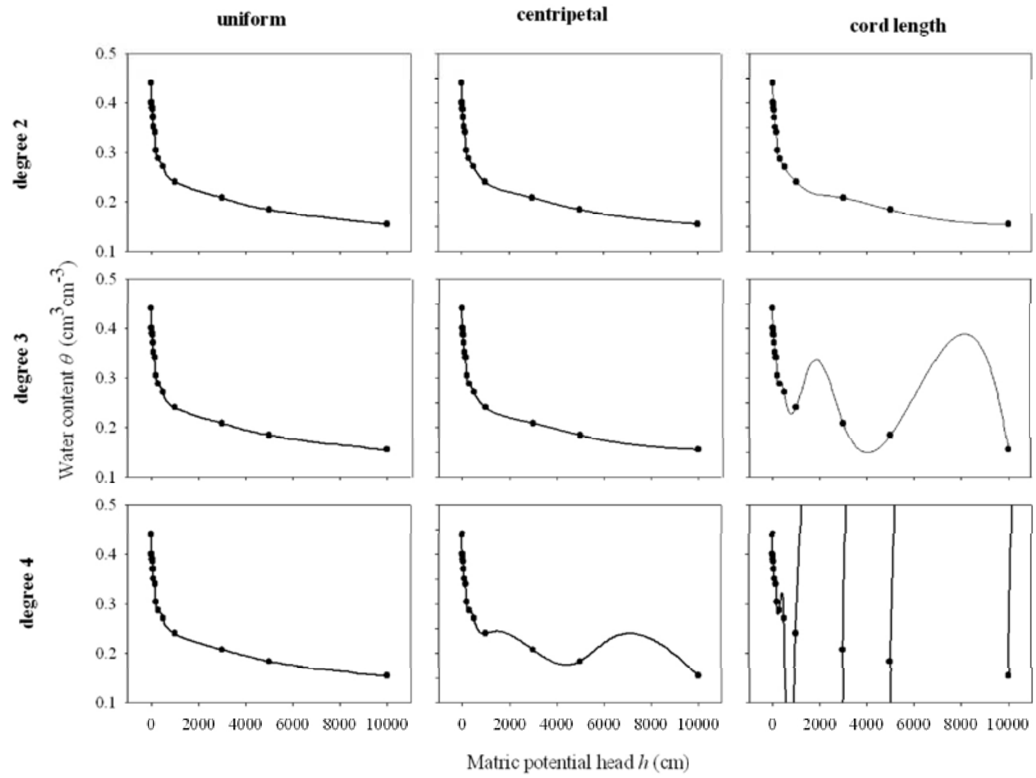

Figure 2:
B-Spline interpolation of degree 2–4 with uniform, cord length and centripetal parametrization
Abbildung 2:
B-Spline Interpolation, 2., 3. und 4. Ordnung mit uniformer, zentripetaler und chord-length Parametrisierung

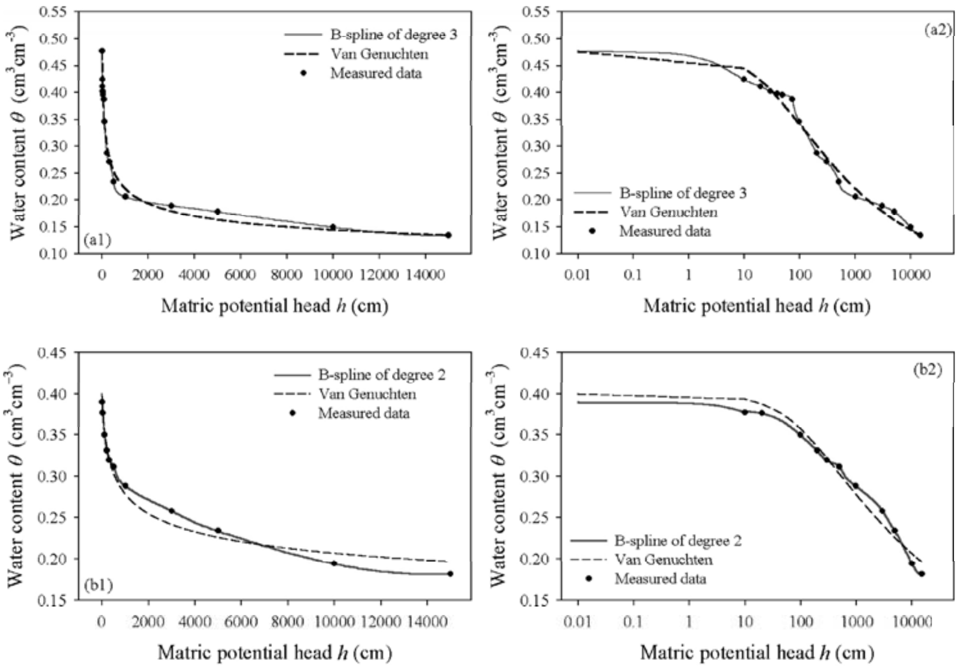

Figure 3:
B-Spline interpolation and van Genuchten approximation curves of (a1) data set CVw.10 in linear and (a2) log scale and (b1) data set NT.10 in linear an (b2) log scale.
Abbildung 3:
B-Spline Interplation und Approximierung mit van Genuchten Kurven der Daten CVw. 10 in (a1) linearer und (a2) logarithmischer Darstellung bzw. der Daten NT.10 in (b) linearer und (b2) logarithmischer Darstellung.

3.3 Results of B-spline Interpolation

In a few cases among the 24 data sets, the shapes of all three spline types considered (B-splines of degree 2 and 3, natural cubic splines) were similarly well (Figure 4). In most of the cases, the natural cubic splines show too many oscillations whereas the B-splines follow the data quite smoothly (see Figure 5).

However, only six out of 24 data sets were truly interpolated with uniform B-splines of degree 2 monotonously, only one with uniform B-spline of degree 3. In most cases, monotony was only violated slightly at the last few data

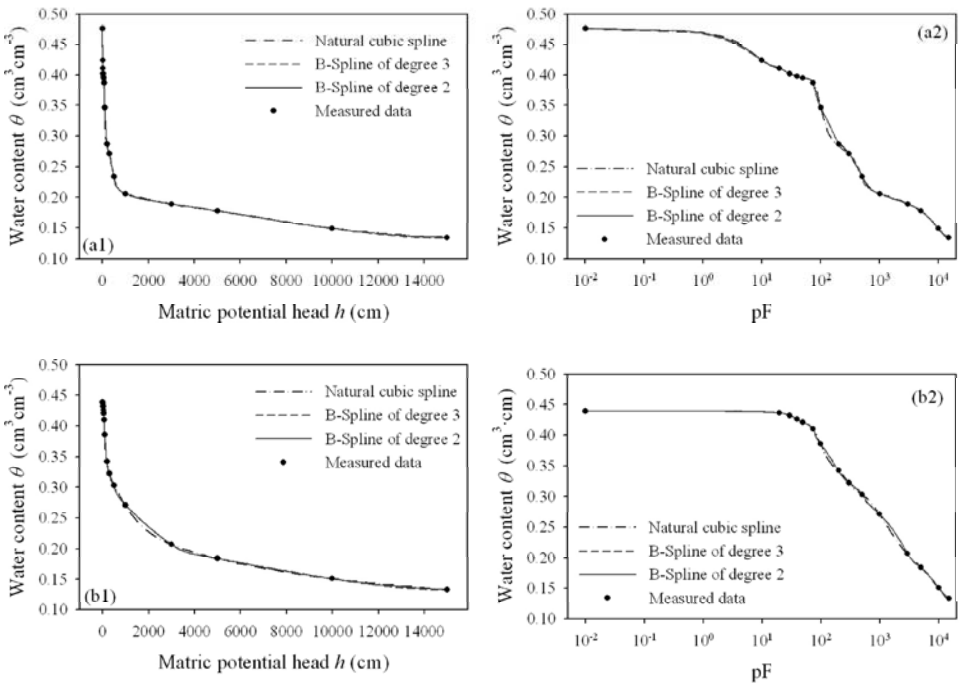

Figure 4:
Interpolation splines of data set CSw.10 in (a1) linear scale, (a2) log scale and CSw.25 in (b1) linear scale, (b2) log scale.
Abbildung 4:
Interpolations-Splines der Daten CSw.10 in (a1) linearer und (a2) logarithmischer Darstellung bzw. der Daten CSw.25 in (b1) linearer und (b2) logarithmischer Darstellung

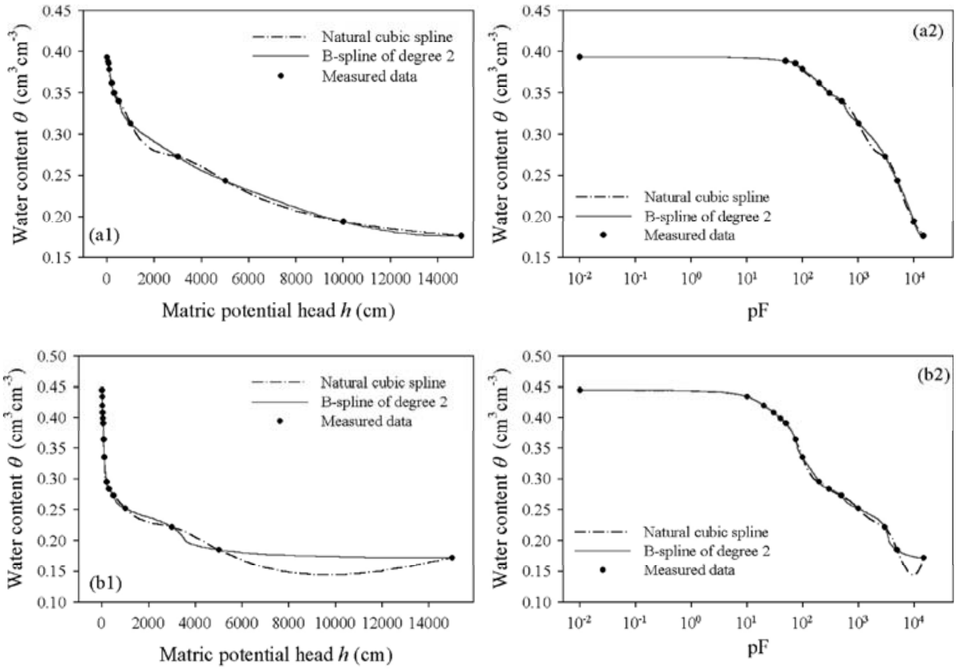

Figure 5:
Interpolation splines of data set NT.50 in (a1) linear scale, (a2) log scale and CSw.25 in (b1) linear scale, (b2) log scale.
Abbildung 5:
Interpolations-Splines der Daten NT.50 in (a1) linearer und (a2) logarithmischer Darstellung bzw. der Daten CSw.25 in (b1) linearer und (b2) logarithmischer Darstellung

points at high matric potential head values. From visual judgment, the curves appeared to follow the data points nicely. However, B-splines do not guarantee that the interpolation curves will be monotonous, an indispensable feature of soil water characteristics. Therefore, there always has to be a certain control by the user when using B-splines. The advantage of B-splines is that they can be modified locally which enables the user to improve slight insufficiencies of the curves by moving a control point.

3.4 Moving control points of B-splines

The local modification property can be used by changing the position of a control point. If control point P_l is moved to P_l+v, then the curve segment on $[u_l, u_{l+p+1})$ is shifted in the direction of v. An example is shown in Figure 6.

Figure 7 shows the interactive window of the JAVA 2 program where the possibility to move control points by mouse drags has been implemented. The inherent feature of an interpolation method is that the resulting curve passes through all data points, which might not be recommended for data containing errors. Approximation techniques are available that relax this requirement (SALOMON 2005). Note that moving control points of an interpolating B-spline curve also results in a curve that is no longer interpolating but approximating.

4 Conclusions

We demonstrated how B-spline curves with different parametrization methods can be used for interpolation of soil water characteristic data. Although the curves generally

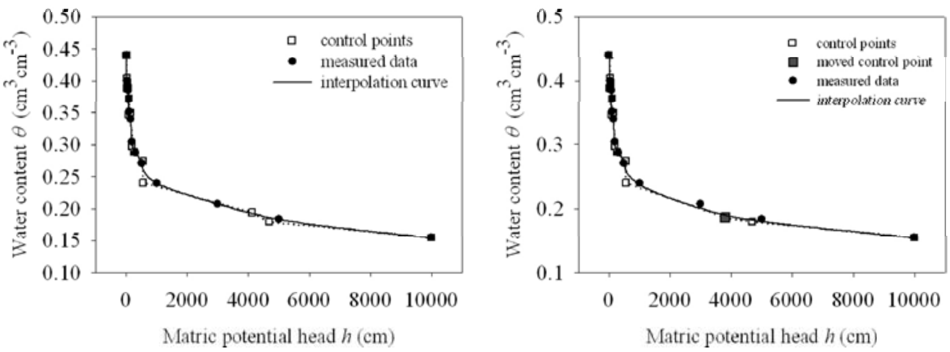

Figure 6:
B-spline of data set NT.50 (a) before and (b) after moving a control point.
Abbildung 6:
B-Sline durch die Daten NT.450 (a) vor und (b) nach Versetzten eines Kontrollpunktes.

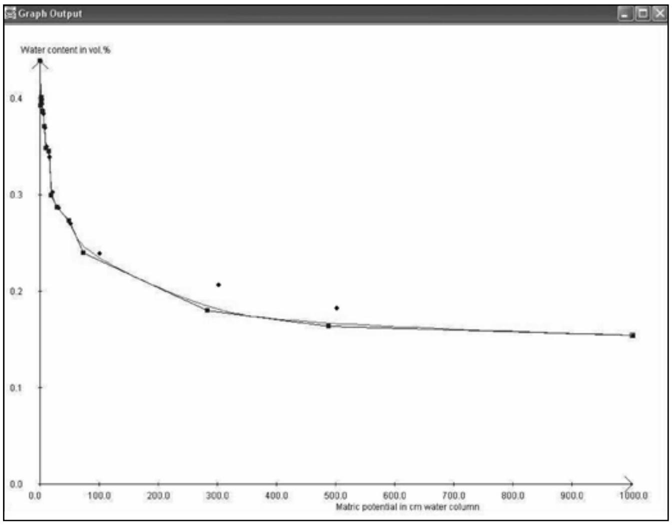

Figure 7: Screen shot of interactive window were control points can be moved by mouse drags (●: measured data, ■ : control points)

Abbildung 7: Abbildung des interaktiven Fensters, in dem die Kontrollpunkte per Mausklick verschoben werden können.

behave well from visual judgment, true monotony cannot be guaranteed. Therefore, manual treatment of resulting interpolation curves may be necessary. We presented a computer program to demonstrate how to do this by moving control points, and thus make use of the local support property of B-splines, with mouse drags. We conclude that, if empirical equations can not be approximated to soil water characteristic data, B-splines are a helpful alternative.

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Appendix

In this appendix, we give a computation example for data interpolation using B-spline curves. Suppose we wish to interpolate the data points shown in Table 1 and Figure 8 using global interpolation with B-splines of degree p=2, with uniform parametrization and clamped knot vector.

Because we have 9 data points, we have that n of Section 2.3.1 is n=8.

Table 1: Data points for computation example
Tabelle 3: Datenpunkte des Berechnungsbeispiels

Matric potential head h	Water content θ
0	0.50421379
10	0.314512505
30	0.237948341
50	0.226827289
100	0.201746862
300	0.169666923
500	0.155212653
1000	0.123261098
1500	0.112785195

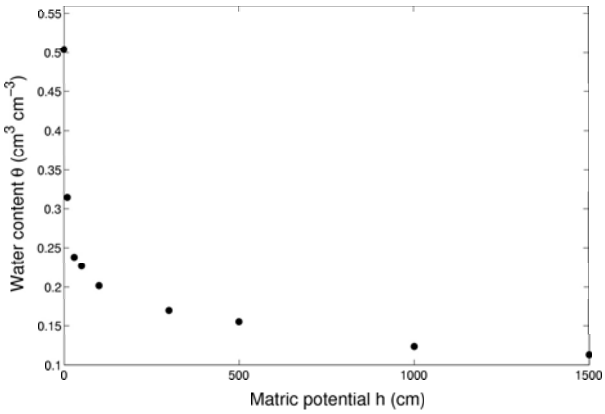

Figure 8: Graph of data points for computation example.
Abbildung 8: Graphische Darstellung der Datenpunkte des Berechnungsbeispiels

To compute the uniformly spaced parameter vector $t=(t_0, t_1, ..., t_n)$, we divide the domain [0, 1500] into equal subintervals,

$$\begin{aligned} t_0 &= 0, \\ t_k &= k\frac{1500}{n}, \quad \text{for} \quad 1 \le k \le n-1, \\ t_n &= 1500. \end{aligned} \tag{10}$$

The resulting parameter vector for this example is given in Table 2.

Table 2: Parameter vector
Tabelle 2: Parametervektor

t_0	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
0.0	187.5	375.0	562.5	562.5	937.5	1125.0	1312.5	1500.0

For the computation of the knot vector u, we use Eq. (6), where $m=n+p+1=11$. The resulting knot vector is given in Table 3.

For each of the entries in the parameter vector, we now compute the B-spline basis functions as in Eq. (2), using the knot vector of Table 3. For example, let us consider the parameter t_5=937.5. It lies in the interval $[u_6, u_7)$= [843.75, 1031.25), and from Eq. (2), we see that

$$\begin{aligned} N_{6,0}(937.5) &= 1, \\ N_{6,1}(937.5) &= \frac{937.5-843.75}{1031.25-843.75}N_{6,0} = \frac{1}{2}, \\ N_{6,2}(937.5) &= \frac{937.5-843.75}{1218.75-843.75}N_{6,1} = 0.125. \end{aligned}$$

Note that in this case, the second term in the sum of Eq. (2) is equal to zero, because 937.5 does not lie in the interval [1031.25,1218.75). Repeating this for every entry of the parameter vector and for every knot span, we obtain the B-spline basic functions

$$N = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.111 & 0.622 & 0.267 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.1 & 0.775 & 0.125 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.125 & 0.75 & 0.125 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.125 & 0.75 & 0.125 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.125 & 0.75 & \mathbf{0.125} & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.125 & 0.775 & 0.1 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.267 & 0.622 & 0.111 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix},$$

where the example which was just calculated is marked in a bold font.

The control points are computed by solving the system of linear equations, Eq. (9), *i.e.* $P=N^{-1} D$. In our example, the control points are

$$P = \begin{pmatrix} 0.0 & 0.504 \\ 3.071 & 0.317 \\ 30.334 & 0.229 \\ 49.470 & 0.230 \\ 72.848 & 0.203 \\ 313.445 & 0.166 \\ 446.485 & 0.159 \\ 1147.935 & 0.110 \\ 1500.0 & 0.113 \end{pmatrix}.$$

Table 3: Knot vector
Tabelle 3: Knotenvektor

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}
0.0	0.0	0.0	281.25	468.75	656.25	843.75	1031.25	1218.75	1500.0	1500.0	1500.0

The points on the B-spline curve between the data points are computed using de Boor's algorithm (GALLIER 1999). In the following, we present its outline as shown at http://en.wikipedia.org/wiki/De_Boor_algorithm (2005):

Input: a value *u*
Output: the point on the curve, *B(u)*

Let *u* lie in $[u_k, u_{k+1})$, with $u \neq u_k$, and let *p* be the degree of the B-spline curve;
Copy the affected control points $P_k, P_{k-1}, P_{k-2}, \ldots, P_{k-p+1}$ and P_{k-p} to a new array and rename them
$P_{k,0}, P_{k-s-1,0}, P_{k-2,0}, \ldots, P_{k-p+1,0}$ and $P_{k-p,0}$;
for $r := 1$ to p do
for $i := k-p+r$ to k do
begin
Let $a_{i,r} = (u - u_i) / (u_{i+p-r+1} - u_i)$
Let $P_{i,r} = (1 - a_{i,r}) P_{i-1,r-1} + a_{i,r} P_{i,r-1}$
end
$P_{k,p}$ is the point *B(u)*.

Suppose we wish to calculate the point on the B-spline curve that corresponds to u=164.4518. This value lies in the interval $[u_2, u_3)$=[0.0, 281.25); remember that the degree p=2. The affected control points are P_0, P_1 *and* P_2, and we rename them $P_{0,0}$, $P_{1,0}$ *and* $P_{2,0}$. The computation of the point on the curve is illustrated in Fig. 9.

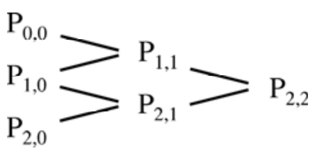

Figure 9: Schematic illustration how the point on the B-spline curve is computed using the affected control points.
Abbildung 9: Schematische Darstellung der Berechnung eines Punktes auf der B-Spline-Kurve unter Verwendung der betroffenen Kontrollpunkte

We compute the second column, $P_{1,1}$ *and* $P_{2,1}$ as follows: The coefficients $a_{1,1}$ and $a_{2,1}$ are

$$a_{1,1} = \frac{u - u_1}{u_{1+2-1+1} - u_1} = \frac{u - u_1}{u_3 - u_1} \approx 0.5847,$$
$$a_{2,1} = \frac{u - u_2}{u_{2+2-1+1} - u_1} = \frac{u - u_2}{u_4 - u_2} \approx 0.3508,$$

and

$$P_{1,1} = (1 - a_{1,1})P_{0,0} + a_{1,1}P_{1,0} \approx (1.7956, 0.3949),$$
$$P_{2,1} = (1 - a_{2,1})P_{1,0} + a_{2,1}P_{2,0} \approx (12.6358, 0.2863).$$

The coefficient for the computation of the third column, $a_{2,2}$, is

$$a_{2,2} = \frac{u - u_2}{u_{2+2-1+1} - u_1} = \frac{u - u_2}{u_4 - u_2} \approx 0.5847,$$

and

$$P_{2,2} = (1 - a_{2,2})P_{1,1} + a_{2,2}P_{2,1} \approx (8.1341, 0.3314).$$

The point (8.13408, 0.33142) is the point on the interpolating B-spline curve (see Fig. 10) that corresponds to u=164.4518.

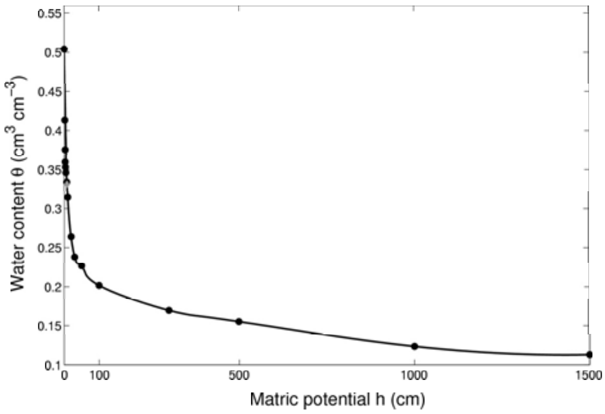

Figure 10: Resulting interpolation B-spline curve. The green diamond marks the point (8.13408, 0.33142)
Abbildung 10: Resultierende B-Spline Interpolationskurve. Der Punkt (8.13408, 0.33142) ist durch eine Raute markiert.